JOURNAL OF APPROXIMATION THEORY 46, 251–283 (1986)

Orthogonal Polynomials with Asymptotically Periodic Recurrence Coefficients*

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Communicated by Paul G. Nevai

Received December 12, 1984; revised February 20, 1985

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Given the coefficients in the three term recurrence relation satisfied by orthogonal polynomials, we investigate the properties of those classes of polynomials whose coefficients are asymptotically periodic. Assuming a rate of convergence of the coefficients to their asymptotic values, we construct the measure with respect to which the polynomials are orthogonal and discuss their asymptotic behavior.

INTRODUCTION

Let μ be a positive measure on the real line with finite moments and infinite support, and let $\{p(x,n)\}_{n=0}^{\infty}$, $p(x,n)=k_n x^n+\dots$, $k_n>0$, be the system of orthonormal polynomials associated with μ. The polynomials $p(x,n)$ satisfy the recurrence formula

$$a(n+1)\,p(x,n+1)+b(n)\,p(x,n)+a(n)\,p(x,n-1)=xp(x,n),$$
$$p(x,-1)=0, \qquad p(x,0)=1.$$

* This work was done while both authors were visiting the Division of Theoretical Physics at Saclay, France.

[†] Supported in part by a NATO Postdoctoral fellowship and NSF Grant DMS-8401609.

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0021-9045/86 $3.00

Recently there has been a great deal of interest [1–13] in investigating the nature of the relationship between the orthogonal polynomials, the recurrence coefficients, and the measure.

Here we investigate the consequences of assuming that the coefficients in the recurrence formula are asymptotically periodic (see (III.3)). This problem is an old one and certain aspects of it were considered by Stieltjes [14], Perron [15], and later by Geronimus [16–18] (see [18] for further references to the Russian literature).

We proceed as follows: in Section II we consider the case where the coefficients in the recurrence formula form periodic sequences. The Green's function (see also Geronimus [17]) is constructed and its analytic properties discussed. Using the Green's function we construct a function which is conformal in a neighborhood of ∞ and maps ∞ to 0 (see Szegö [19, Chap. XVI] and Barnsley, Geronimo, and Harrington [20]). Then (Sect. III) we consider the general case as a perturbation of the periodic case. The techniques of scattering theory are introduced and used to investigate the properties of the general system when it is assumed that the coefficients converge to their asymptotic values at certain preassigned rates. In Section IV the measure, with respect to which the polynomials satisfying the recurrence formula are orthonormal, is constructed and the properties of the measure are investigated. It is shown that under certain conditions, it falls into the Szegö class. Finally in Section V we investigate the asymptotic behavior of the orthogonal polynomials.

II. The Periodic Case

Given $a_0(n+1)>0$ and $b_0(n)\in\mathbb{R}$ for $n=0, 1, 2,...$ and assuming the periodicity condition

$$\begin{aligned} a_0(n+N)&=a_0(n), \qquad n=1, 2,..., \\ b_0(n+N)&=b_0(n), \qquad n=0, 1, 2,..., \end{aligned} \tag{II.1}$$

$N\geqslant 1$, we form the following three term recurrence formula

$$\begin{aligned} &a_0(n+1)\, q(x, n+1)+b_0(n)\, q(x, n) \\ &\quad + a_0(n)\, q(x, n-1)=xq(x, n), \qquad n=0, 1, 2,... \end{aligned} \tag{II.2}$$

(here we take $a_0(0)=a_0(N)$). If we impose the boundary condition

$$q(x, 0)=1, \qquad q(x, -1)=0, \tag{II.3}$$

then $q(x, n)$ is a polynomial of degree n with leading coefficient positive and we know from Favard [21], that there exists a probability measure μ_0 such that

$$\int_{-\infty}^{\infty} q(x, n)\, q(x, m)\, d\mu_0 = \delta_{n,m}. \tag{II.4}$$

We can also associate with the above coefficients the kth associated polynomials $q^{(k)}(x, n)$ satisfying

$$\begin{gathered} a_0(n+k+1)\, q^{(k)}(x, n+1) + b_0(n+k)\, q^{(k)}(x, n) + a_0(n+k)\, q^{(k)}(x, n-1) \\ = xq^{(k)}(x, n), \qquad n = 0, 1, 2, \ldots, \\ q^{(k)}(x, 0) = 1, \qquad q^{(k)}(x, -1) = 0. \end{gathered} \tag{II.5}$$

Given any two solutions q_1, q_2 of (II.2), we define the Wronskian $W[q_1, q_2]$ as

$$W[q_1, q_2] = a_0(n+1)\{q_1(x, n+1)\, q_2(x, n) - q_1(x, n)\, q_2(x, n+1)\}, \tag{II.6}$$

which is independent of n. Furthermore from the general theory of second-order linear difference equations one finds that two solutions q_1, q_2 of (II.2) are linearly independent in n iff $W[q_1, q_2] \neq 0$.

As a first application of (II.6) we notice

$$\begin{aligned} W[q, q^{(1)}] &= a_0(n+1)[q(x, n+1)\, q^{(1)}(x, n-1) - q(x, n)\, q^{(1)}(x, n)] \\ &= -a_0(1) \neq 0, \end{aligned} \tag{II.7}$$

which implies that $q(x, n)$ and $q^{(1)}(x, n)$ are two linearly independent solutions of (II.2).

To investigate the consequences of the periodicity condition (II.1) we begin by constructing a recurrence relation that relates $q(x, n+2N)$ and $q(x, n+N)$ to $q(x, n)$.

LEMMA 1. *Let* $q_1(x, n)$ *be any solution of* (II.2) *and let the recurrence coefficients satisfy* (II.1), *then*

$$\begin{aligned} q_1(x, n+2N) = \left\{ q(x, N) - \frac{a_0(N)}{a_0(N+1)}\, q^{(1)}(x, N-2) \right\} q_1(x, n+N) \\ - q_1(x, n), \qquad n = 0, 1, 2, \ldots. \end{aligned} \tag{II.8}$$

Proof. Because of the periodicity of the coefficients, we see that $q(x, n+N)$ and $q^{(1)}(x, n+N-1)$ will again be solutions of (II.2) so that,

$$q(x, n+N) = Aq(x, n) + Bq^{(1)}(x, n-1) \tag{II.9}$$

$$q^{(1)}(x, n+N-1) = Cq(x, n) + Dq^{(1)}(x, n-1) \tag{II.10}$$

where A, B, C, and D do not depend on n. Setting n equal to 0 and -1 we find

$$A = q(x, N), \qquad B = -\frac{a_0(N)}{a_0(N+1)}\, q(x, N-1),$$
$$C = q^{(1)}(x, N-1), \qquad D = -\frac{a_0(N)}{a_0(N+1)}\, q^{(1)}(x, N-2). \tag{II.11}$$

Letting $n \to n+N$ in (II.9) then eliminating $q^{(1)}(x, n+N-1)$ using (II.10) and $q^{(1)}(x, n-1)$, using (II.5) yields

$$q(x, n+2N) = (A+D)\, q(x, n+N) + (BC - AD)\, q(x, n).$$

By means of (II.7) one finds that $BC-AD = -1$ and this coupled with (II.11) gives (II.8) for q. Using a similar procedure on (II.10) one arrives at (II.4) for $q^{(1)}$ and since all the solutions of (II.2) can be written as a linear combination of q and $q^{(1)}$ the result follows.

COROLLARY 1. *Given* (II.1) *and* $q^{(k)}(x, n)$, $k \geqslant 0$ *satisfying* (II.5) *one has for* $k \geqslant 0$,

$$q^{(k)}(x, N) - \frac{a_0(N+k)}{a_0(N+k+1)}\, q^{(k+1)}(x, N-2)$$
$$= q(x, N) - \frac{a_0(N)}{a_0(N+1)}\, q^{(1)}(x, N-2).$$

Proof. The polynomials $q^{(k)}(x, n)$ satisfy a recurrence relation with periodic coefficients and since $q^{(k)}$ and $q^{(k+1)}$ are linearly independent solutions of (II.5) one finds

$$q^{(k)}(x, n+2N) = \left\{ q^{(k)}(x, N) - \frac{a_0(N+k)}{a_0(N+k+1)}\, q^{(k+1)}(x, N-2) \right\}$$
$$\times\, q^{(k)}(x, n+N) - q^{(k)}(x, n), \qquad n = 0, 1, 2, \ldots.$$

On the other hand $q^{(k)}(x, n-k)$ also satisfies (II.2) and it is a consequence of Lemma 1 with n replaced by $n+k$ that $q^{(k)}(x, n)$ satisfies (II.8). The identification of both relations then gives the corollary.

Remark 1. Relation (II.8) is again a recurrence relation but with coefficients constant in n. Certain solutions of this equation will play a fundamental role in what is to follow.

Applying the method of characteristic equations to (II.8) we find

$$\alpha^{2N}-\left\{q(x, N)-\frac{a_0(N)}{a_0(N+1)}\,q^{(1)}(x, N-2)\right\}\alpha^N+1=0. \tag{II.12}$$

Splitting the above equation into two equations of degree N, we define

$$w^N(x)=w^N=\frac{1}{2}\left\{q(x, N)-\frac{a_0(N)}{a_0(N+1)}\,q^{(1)}(x, N-2)+\rho(x)\right\} \tag{II.13}$$

where

$$\rho(x)=\left\{\left(q(x, N)-\frac{a_0(N)}{a_0(N+1)}\,q^{(1)}(x, N-2)\right)^2-4\right\}^{1/2} \tag{II.14}$$

with the square root chosen such that

$$\lim_{z\to\infty}\frac{\rho(z)}{z^N}=k_N=\prod_{i=1}^{N}\frac{1}{a_0(i)}>0.$$

Since the constant term in (II.12) is one, we have

$$w^{-N}=\frac{1}{2}\left\{q(x, N)-\frac{a_0(N)}{a_0(N+1)}\,q^{(1)}(x, N-2)-\rho(x)\right\}. \tag{II.15}$$

We now examine $\rho(x)^2$; setting

$$Q_{\pm}(x, N)=q(x, N)-\frac{a_0(N)}{a_0(N+1)}\,q^{(1)}(x, N-2)\pm 2 \tag{II.16}$$

we denote the zeros of $Q_{\pm}(x, N)$ by $\{x_i^{\pm}\}_{i=1}^{N}$. Let $x_{j,N-1}$ and $x_{j,N-1}^{(1)}$ be the zeros of $q(x, N-1)$ and $q^{(1)}(x, N-1)$ respectively, ordered so that $x_{j,N-1}<x_{j+1,N-1}$, and $x_{j,N-1}^{(1)}<x_{j+1,N-1}^{(1)}$, $j=1, 2,..., N-2$.

LEMMA 2 (cf. Geronimus [16], Kac and Van Moerbeke [24], Van Moerbeke [25]). *All the zeros of $\rho(x)^2$ are real (but not necessarily simple) and, ordering the zeros of $Q_{\pm}(x)$ as $x_i^{\pm}\leqslant x_{i+1}^{\pm}$ one has,*

$$\begin{gathered} x_N^- > x_N^+ \geqslant x_{N-1,N-1},\ x_{N-1,N-1}^{(1)} \geqslant x_{N-1}^+ > x_{N-1}^- \geqslant x_{N-2,N-1}, \\ x_{N-2,N-1}^{(1)} \geqslant \cdots \geqslant x_{1,N-1},\ x_{1,N-1}^{(1)} \geqslant x_1^{(-)^N} > x_1^{(-)^{N+1}}. \end{gathered} \tag{II.17}$$

Furthermore, if $|q(x_{i,N-1}, N)|=1$, then either $Q_+(x, N)$ or $Q_-(x, N)$ may have a double zero.

Proof. By means of (II.7) one finds that,

$$a_0(N)\, q(x_{j,N-1}, N)\, q^{(1)}(x_{j,N-1}, N-2) = -a_0(N+1), \qquad \text{(II.18)}$$

so that

$$Q_\pm(x_{j,N-1}, N) = q(x_{j,N-1}, N) + \frac{1}{q(x_{j,N-1}, N)} \pm 2. \qquad \text{(II.19)}$$

Since the zeros of $q(x, N)$ and $q(x, N-1)$ interlace one finds that $\operatorname{sgn} q(x_{j,N-1}, N) = (-1)^{N-j}$. The above remarks coupled with the fact that $|x+1/x| \geqslant 2$ for x real imply that $Q_\pm(x, N)$ change sign $N-1$ times, which in turn implies, since they are real polynomials, the reality of their zeros and the interlacing of their zeros with those of $q(x, N-1)$. Since the zeros of $q^{(1)}(x, N-1)$ and $q(x, N)$ interlace an argument similar to the one above gives the result for $q^{(1)}(x, N-1)$. To arrive at (II.17) we note that for x large

$$q(x, N) - \frac{a_0(N)}{a_0(N+1)}\, q^{(1)}(x, N-2)$$

is positive, which implies for large enough x, $Q_\pm(x, N) > 0$. Consequently $x_N^+ < x_N^-$. At $x_{N-1,N-1}$, $Q_\pm(x_{N-1,N-1}, N) \leqslant 0$ so that $Q_+(x, N-1)$ must have a zero greater than or at $x_{N-1,N-1}$. Since $Q_+(x, N) - Q_-(x, N) = 4$, the next zero of $Q_\pm(x, N)$ is a zero of $Q_+(x, N)$ which, because of the interlacing property, will be before $x_{N-2,N-1}$ and on or after $x_{N-1,N-1}$. At $x_{N-2,N-1}$, $Q_\pm(x_{N-2,N-1}, N) \geqslant 0$ so that $x_{N-2,N-1} \leqslant x_{N-1}^- < x_{N-1}^+$. This establishes (II.17). If $|q(x_{j,N-1}, N)| = 1$ for some j, then from (II.19), $x_{j,N-1}$ will be a zero of either $Q_+(x, N)$ or $Q_-(x, N)$ and we see from (II.17) that $Q_\pm(x, N)$ may have a double zero.

We now define the set E:

$$E = [x_N^+, x_N^-] \cup [x_{N-1}^-, x_{N-1}^+] \cup \cdots \cup [x_1^{(-)^{N+1}}, x_1^{(-)^N}] \qquad \text{(II.20)}$$

which is composed of at most N disjoint intervals, the set E^*:

$$E^* = (x_{N-1}^+, x_N^+) \cup (x_{N-2}^-, x_{N-1}^-) \cup \cdots \cup (x_1^{(-)^N}, x_2^{(-)^N}), \qquad \text{(II.20A)}$$

and the polynomial $U(x, N-1) = Q'_+(x, N) = Q'_-(x, N)$. From the above definitions, it is obvious that in each of the above open intervals defining E^* that is not empty there will be one and only one zero of $U(x, N-1)$ (see Fig. 1).

Let U be the unit circle, D be the open unit disk, and $\bar{D} = D \cup U$. Let $\hat{\mathbb{C}}$ be the extended complex plane and $G = \hat{\mathbb{C}} \backslash E$. Let $g(z)$ be the Green's function for G, that is $g(z)$ is harmonic in G except at ∞ where $g(z) - \ln|z|$

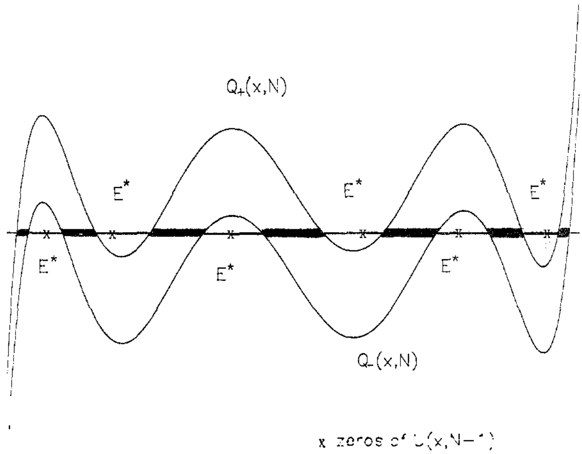

FIG. 1. Construction of E and E^*. The E is in bold line, the set E^* are the intervals in between ($N = 7$).

is harmonic, and $\lim_{z \to E} g(z) = 0$. We now form the function $\hat{g}(z)$ by adding ih, to $g(z)$, h a conjugate harmonic function of $g(z)$ chosen so that $\hat{g}(z)$ is a multivalued analytic function on G except at ∞, where $\hat{g}(z) - \ln z$ is analytic. $\hat{g}(z)$ has the property that $\lim_{z \to E} \operatorname{Re} \hat{g}(z) = 0$. In our case one sees that

$$\lim_{z \to \infty} \left| \frac{w(z)}{z} \right| = k_N^{1/N}$$

and $|w(z)| = 1$, $z \in E$ and therefore we may choose an appropriate branch of the Nth root so that $\hat{g}(z) = \ln w(z)$. Thus, the capacity of E is given by

$$C(E) = \left(\prod_{i=1}^{N} a_0(i) \right)^{1/N} > 0.$$

Since $|w(z)| = 1$, $z \in E$ we see that $w(z)$ maps G into the component of the complement of U containing ∞. However, because G is in general not a simply connected region, $w(z)$ is in general not single valued. For large enough z, $w(z)$ is conformal and we let γ be the inverse of w. For each $\theta \in [0, 2\pi)$ we define $r_0(\theta)$ to be the minimum number $\geqslant 1$ such that γ may be analytically continued from ∞ along $R_\theta = \{re^{i\theta} \mid r > r_0\}$ (physically they are the lines of force). The set $s = \bigcup_\theta \gamma(R_\theta)$ is called the Green's star domain for G (see Sario and Nakai [22]). In our case

$$w'(z) = \frac{w(z)\, U(z, N-1)}{N\rho(z)}$$

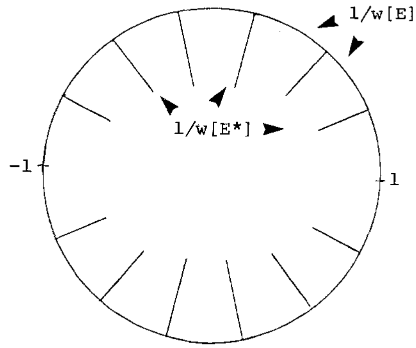

FIG. 2. The image of G under the mapping $1/w$ ($N=7$).

and $w'(z)=0$ ($z \in G$) at the zeroes of $U(z, N-1)$ that are in E^*. Consequently $s=(E \cup E^*)^c$. On s, $w(z)$ is conformal and maps s to the exterior of U minus radial segments emanating from the roots of unity given by (II.13) and ending at the image of one of the zeros of $U(z, N-1)$ under $w(z)$. In Fig. 2, we have drawn $1/w(s)$.

Setting

$$R(z)=\left(q(z, N)-\frac{a_0(N)}{a_0(N+1)}\, q^{(1)}(z, N-2)\right)^2-4,$$

we have by the convention adopted above, $\sqrt{R(x \pm i0)}=\pm i\sqrt{-R(x)}$, $x \in E$. Let F be the two sheeted Riemann surface which has cuts along the disconnected segments E with branch points at the ends of these segments. Then F is of genus at most $N-1$ and G is one sheet of F. Denoting the other sheet as G' one has that

$$\lim_{z \to \infty} \frac{\rho(z)}{z^N}=-k_N \qquad \text{on } G'.$$

With this we can now analytically continue w^N and w^{-N} onto G'.

We now return to the solutions of (II.5).

LEMMA 3. *Let* $q^{(i)}(x, n)$ *satisfy* (II.5) *then*

$$|w^{-n} q^{(i)}(x, n)| \leqslant \frac{A(n+N)}{N+|1-w^{-2N}|\,(n+N)} \tag{II.21}$$

where A is a positive constant.

Proof. From Lemma 1 and Corollary 1, the sequence $q^*(k)= q^{(i)}(x, kN+s)$ (i, N, s fixed) satisfies the relation

$$q^*(k+1)=\left\{q(x, N)-\frac{a_0(N)}{a_0(N+1)}\, q^{(1)}(x, N-2)\right\} q^*(k)-q^*(k-1).$$

Since w^{-kN} and w^{kN} are two linearly independent solutions of the above equation when $\rho(x) \neq 0$, one has

$$q^*(k) = C_1 w^{kN} + C_2 w^{-kN}.$$

Setting $k = 0$ and $k = 1$, one easily finds,

$$w^{-(kN)} q^{(i)}(x, kN+s) = \frac{1}{1 - w^{-2N}} \{ q^{(i)}(x, s)(w^{-2kN} - w^{-2N}) + w^{-(N)} q^{(i)}(x, N+s)(1 - w^{-2kN}) \}. \tag{II.22}$$

Since $\lim_{x \to \infty}(x/w) < \infty$ one has

$$K = \max_{0 \leqslant i < N} \max_{0 \leqslant s < 2N} \sup_{x \in \mathbb{C}} |w^{-s} q^{(i)}(x, s)| < \infty. \tag{II.23}$$

and using this in (II.22) yields

$$|w^{-(kN+s)} q^{(i)}(x, kN+s)| \leqslant \frac{K}{|1 - w^{-2N}|} \{ |w^{-2kN} - w^{-2N}| + |1 - w^{-2kN}| \}.$$

Now one can use the bound

$$\left| \frac{1 - z^{2n}}{1 - z^2} \right| \leqslant C \frac{n}{1 + |1 - z^2|\, n} \qquad (|z| \leqslant 1)$$

to obtain

$$|w^{-(kN+s)} q^{(i)}(x, kN+s)| \leqslant A \frac{k+1}{1 + |1 - w^{-2N}|\,(k+1)}$$

where $A = 2CK$, from which the general bound follows.

THEOREM 1. *Set*

$$q_+(x, n) = q(x, n+N) - w^N q(x, n) \tag{II.24}$$

and

$$q_-(x, n) = q(x, n+N) - w^{-N} q(x, n), \tag{II.25}$$

then $q_\pm(x, n)$ *are two solutions of* (II.2) *such that* $q_\pm(x, n) = w^{\mp n} \phi_\pm(x, n)$, *where* $\phi_\pm(x, n)$ *is periodic in n of period N. These two solutions are linearly independent in n for fixed x iff* $q(x, N-1) \neq 0$ *and* $w^{2N} \neq 1$.

Proof. Since both $q(x, n+N)$ and $q(x, n)$ are solutions of (II.2) one easily sees that $q_\pm(x, n)$ are also solutions. From Lemma 1, one has

$$q(x, n+2N) = (w^N + w^{-N})\, q(x, n+N) - q(x, n). \tag{II.26}$$

Letting $n \to N+n$ in (II.24) and (II.25) then substituting the result into the above equation yields

$$q_\pm(x, n+N) = w^{\mp N} q_\pm(x, n). \tag{II.27}$$

From (II.6) one finds

$$W[q_+, q_-] = a_0(N)\, q(x, N-1)[w^{-N} - w^N] \tag{II.28}$$

which gives the theorem.

LEMMA 4. *For any* n, $q_+(x, n)$ *and* $q_-(x, n)$ *are* (a) *analytic and single valued in* $G - \{\infty\}$, (b) *real for* x *real* $\notin E$, *and* (c) $q_+(x, n) = \overline{q_-(x, n)}$ $x \in E$. *Furthermore,*

$$|w^{-N-n} q_-(x, n)| < 2K, \qquad x \in \hat{\mathbb{C}}, \tag{II.29}$$

and

$$|w^{n-N+2} q_+(x, n)| < D, \qquad x \in \hat{\mathbb{C}}. \tag{II.30}$$

Proof. The analytic properties follow from the definition of $q_\pm$ and the following facts: (a) w^N and w^{-N} are single valued and analytic on $G - \{\infty\}$, (b) w^N and w^{-N} are real for x real $x \notin E$, and (c) $w^N = \bar{w}^{-N}$, $x \in E$. (II.29) follows by writing $n = kN + s$ then using (II.27) and (II.23). To prove (II.30) one has that

$$\begin{aligned} q_+(x, n)\, q_-(x, n) &= q(x, n+N)^2 - (w^N + w^{-N})\, q(x, n+N)\, q(x, n) \\ &\quad + q(x, n)^2 \\ &= q(x, n+N)^2 - q(x, n+2N)\, q(x, n), \end{aligned}$$

where Lemma 1 has been used. Now $q^{(m+1)}(x, n-m-1)$ is a solution of (II.2) and can be written as a linear combination of $q(x, n)$ and $q(x, n+N)$, i.e.,

$$\begin{aligned} q^{(m+1)}(x, n-m-1) = &\frac{a_0(m+1)}{a_0(N)} \frac{1}{q(x, N-1)} \\ &\times [q(x, m+N)\, q(x, n) - q(x, m)\, q(x, n+N)]. \end{aligned}$$

Replacing n by $m+N$ and then setting $m=n$ yields

$$q(x, N-1)\, q^{(n+1)}(x, N-1)=\frac{a_0(n+1)}{a_0(N)} \times \{q(x, n+N)^2-q(x, n+2N)\, q(x, n)\}.$$

Consequently,

$$q_+(x, n)\, q_-(x, n)=\frac{a_0(N)}{a_0(n+1)}\, q(x, N-1)\, q^{(n+1)}(x, N-1). \tag{II.31}$$

Now using the fact that $q_-(x, n)=O(w^{n+N})$, Lemma 3 gives (II.30).

Equation (II.31) leads to the following:

LEMMA 5. *The zeros of* $q_-(x, kN+s)$ *and* $q_-(x, kN+s)$, $k=0, 1,..., s=0, 1,..., N-1$, *are real and may only be at the zeros of* $q(x, N-1)$ *and/or the zeros of* $q^{(s+1)}(x, N-1)$. *Furthermore, a zero* x_j *of* $q(x, N-1)$ *will be a common zero of* $q_+(x, n)$ *for all n if and only if* $|q(x_j, N)| \geqslant 1$.

Proof. Only the last part of the lemma needs to be demonstrated. We note that x_j will be a common zero of $q_-(x, n)$ for all n if and only if $q_+(x_j, 0)=0$. From (II.24) and (II.13), one sees that this can only happen if

$$q(x_j, N)+\frac{a_0(N)}{a_0(N+1)}\, q^{(1)}(x_j, N-2)=\rho(x_j).$$

Since

$$\left(q(x_j, N)+\frac{a_0(N)}{a_0(N+1)}\, q^{(1)}(x_j, N-2)\right)^2=\rho(x_j)^2,$$

we see that $q_+(x_j, 0)=0$ if and only if

$$q(x_j, N)+\frac{a_0(N)}{a_0(N+1)}\, q^{(1)}(x_j, N-2)$$

and $\rho(x_j)$ have the same sign. Since the signs of $q(x_j, N)$ and $\rho(x_j)$ are the same, (II.7) shows that $q_+(x_j, 0)=0$ if and only if $|q(x_j, N)| \geqslant 1$.

We note that from (II.25) and (II.31), one finds for large x that

$$\frac{q_+(x, n)}{q(x, N-1)} \approx x^{n+1} \Big/ \prod_{i=0}^{n} \frac{1}{a_0(i)}. \tag{II.32}$$

III. The General Case

We now suppose we are given the recurrence relation

$$a(n+1)\,p(x,n+1)+b(n)\,p(x,n)+a(n)\,p(x,n-1)$$
$$=xp(x,n), \qquad n=0,1,2,..., \tag{III.1}$$
$$p(x,-1)=0, \qquad p(x,0)=1 \tag{III.2}$$

with $a(n)>0$ and $b(n)\in\mathbb{R}$ such that

$$\lim_{n\to\infty}|a(n)-a_0(n)|=0,$$
$$\lim_{n\to\infty}|b(n)-b_0(n)|=0, \tag{III.3}$$

where the sequences $a_0(n)$ and $b_0(n)$ satisfy (II.1). By Favard's theorem, the polynomials $p(x,n)$ will be orthogonal with respect to some measure on the real line. The kth associated polynomials $p^{(k)}(x,n)$ satisfy

$$a(n+k+1)\,p^{(k)}(x,n+1)+b(n+k)\,p^{(k)}(x,n)$$
$$+a(n+k)\,p^{(k)}(x,n-1)$$
$$=xp^{(k)}(x,n), \qquad n=0,1,2,..., k\geqslant 0. \tag{III.4}$$

(We will suppress the superscript for $k=0$.) Let G_1, G_2 be the solutions of

$$a_0(n+1)\,G_i(x,n+1,m)+b_0(n)\,G_i(x,n,m)+a_0(n)\,G_i(x,n-1,m)$$
$$-\,xG_i(x,n,m)=\delta_{n,m}, \qquad i=1,\ 2, \tag{III.5}$$

with boundary conditions

$$G_1(x,n,m)=0, \qquad n\geqslant m,$$
$$G_2(x,n,m)=0, \qquad n\leqslant m,$$

then (Geronimo [9], Atkinson [23])

$$a_0(n+1)\,G_1(x,n,m)=q^{(n+1)}(x,m-n-1), \qquad -1\leqslant n<m,$$
$$=0, \qquad m\leqslant n, \tag{III.6}$$

and

$$a_0(m+1)\,G_2(x,n,m)=q^{(m+1)}(x,n-m-1), \qquad -1\leqslant m<n,$$
$$=0, \qquad n\leqslant m. \tag{III.7}$$

THEOREM 2. *Let $p(x, n)$ satisfy* (III.1) *and* (III.2) *and let*

$$\hat{p}(x, n) = \prod_{i=1}^{n} \frac{a(i)}{a_0(i)} p(x, n),$$

then

$$\hat{p}(x, m) = q(x, m) + \sum_{n=0}^{m-1} k_1(x, n, m) \hat{p}(x, n) \tag{III.8}$$

where

$$\begin{aligned} k_1(x, n, m) = {} & \{b_0(n) - b(n)\} \, G_1(x, n, m) \\ & + a_0(n+1) \left\{ 1 - \frac{a^2(n+1)}{a_0^2(n+1)} \right\} G_1(x, n+1, m). \end{aligned} \tag{III.9}$$

Furthermore, for $x \in \hat{\mathbb{C}}$,

$$\begin{aligned} |w^{-m} \hat{p}(x, m)| \leqslant {} & \frac{A(m+N)}{N + |1 - w^{-2N}| \, (m+N)} \\ & \times \exp\left\{ A \sum_{n=0}^{m-1} k(n) \frac{n+N}{N + |1 - w^{-2N}| \, (n+N)} \right\} \end{aligned} \tag{III.10}$$

where A is a positive constant and

$$k(n) = \left| \frac{b_0(n) - b(n)}{a_0(n+1)} \right| + \frac{a_0(n+1)}{a_0(n+2)} \left| 1 - \frac{a^2(n+1)}{a_0^2(n+1)} \right|. \tag{III.11}$$

Proof. Equation (III.8) has already been given in Geronimo [9] and can easily be derived using standard manipulations. To obtain the bound (III.10) one begins by substituting (III.6) into (III.9) then multiplying by $|w|^{-(m-n)}$ and using Lemma 3 which yields

$$\begin{aligned} |w^{-(m-n)} k_1(x, n, m)| & \leqslant A k(n) \frac{m - n + N}{N + |1 + w^{-2N}| \, (m - n + N)} \\ & \leqslant A k(n) \frac{m + N}{N + |1 - w^{-2N}| \, (m+N)}. \end{aligned} \tag{III.12}$$

Now using the method of successive approximations on (III.8) we may write

$$|w|^{-m} \hat{p}(x, m) = \sum_{i=0}^{\infty} g_i(x, m) \tag{III.13}$$

where

$$g_0(x, m) = |w|^{-m} q(x, m)$$

and

$$g_i(x, m) = \sum_{n=0}^{m-1} |w|^{-(m-n)} k_1(x, n, m) g_{i-1}(x, n).$$

From Lemma 3, one has that

$$|g_0(x, m)| \leqslant \frac{A(m+N)}{N + |1 - w^{-2N}| (m+N)}$$

and by induction that

$$|g_i(x, m)| \leqslant \frac{A(m+N)}{N + |1 - w^{-2N}| (m+N)} \frac{1}{i!} \left\{ A \sum_{n=0}^{m-1} \frac{k_1(n)(n+N)}{N + |1 - w^{-2N}| (n+N)} \right\}^i.$$

Taking the magnitude of (III.13) then using the above two equations gives (III.10).

We now search for solutions $p_{\pm}(x, n)$ such that

$$\lim_{n \to \infty} |p_{\pm}(x, n) - q_{\pm}(x, n)| = 0.$$

To this end we temporarily impose the following condition on coefficients

$$\begin{aligned} a(n_0 + j + 1) &= a_0(n_0 + j + 1) \\ b(n_0 + j) &= b_0(n_0 + j) \qquad j = 0, 1, 2, \ldots. \end{aligned} \tag{III.14}$$

We denote the solutions of (III.1), (III.2), and (III.14) by $p(x, m; n_0)$ and we define $p_{+}(x, m; n_0)$ as a solution of (III.1) such that $p_{\pm}(x, m; n_0) = q_{\pm}(x, m)$ for $m \geqslant n_0$.

Lemma 6. *Let $p_{+}(x, m; n_0)$ be defined as above and set*

$$\hat{p}_{+}(x, m; n_0) = \prod_{i=m+1}^{\infty} \frac{a(i)}{a_0(i)} p_{+}(x, m; n_0)$$

then

$$\hat{p}_{+}(x, m; n_0) = q_{+}(x, m) + \sum_{n=m+1}^{n_0} k_2(x, n, m) \hat{p}_{+}(x, n; n_0) \tag{III.15}$$

where

$$k_2(x, n, m) = \{b_0(n) - b(n)\}\, G_2(x, n, m) + a_0(n)\left\{1 - \frac{a^2(n)}{a_0^2(n)}\right\} G_2(x, n-1, m). \tag{III.16}$$

(*Here* $\sum_{n=i}^{j} f_n \equiv 0$ *for* $i > j$.)

Proof. To find (III.15) one begins by multiplying (III.1) by

$$\prod_{i=n+1}^{\infty} \frac{a(i)}{a_0(i)}\, G_2(x, n, m)$$

and (III.5) by $\hat{p}_+(x, n, n_0)$ then subtracting and summing the result from 0 to ∞. This yields using the appropriate boundary condition,

$$\begin{aligned}\hat{p}_+(x, m; n_0) &= a_0(n_0+1)\{q_+(x, n_0)\, G_2(x, n_0+1, m) - q_+(x, n_0+1)\, G_2(x, n_0, m)\} \\ &+ \sum_{n=m+1}^{n_0} k_2(x, n, m)\, \hat{p}_+(x, n; n_0).\end{aligned} \tag{III.19}$$

The first term on the right-hand side is $W[G_2, q_+]$ which equals $q_+(x, m)$ and gives (III.15).

THEOREM 3. *Let* $H = G \cup \partial G$ *and suppose*

$$\sum_{n=0}^{\infty} k_2(n) < \infty \tag{III.20}$$

where

$$k_2(n) = |b_0(n) - b(n)| + a_0(n+1)\left|1 - \frac{a^2(n+1)}{a_0^2(n+1)}\right|,$$

then there exists a solution $p_+(x, n)$ *of* (III.1) *such that* $w^{-N} p_+(x, m)$ *is analytic and single valued on G and continuous on* $H \backslash (w^{2N} = 1)$. *Furthermore*

$$\lim_{m \to \infty} |w^{(k-1)N}(\hat{p}_+(x, m) - q_+(x, m))| = 0, \qquad m = kN + s, \tag{III.21}$$

uniformly on closed subsets of $H \backslash (w^{2N} = 1)$. *If*

$$\sum_{n=0}^{\infty} (n + N)\, k_2(n) < \infty \tag{III.22}$$

then $w^{-N} p_+(x, n)$ *is continuous on H and* (III.21) *converges uniformly there.*

Proof. We begin formally by letting $n_0 \to \infty$ in (III.15). This gives us an integral equation for $p_+(x, n)$. Now using the method of successive approximations we write

$$w^{(k-1)N}\hat{p}_+(x, m) = \sum_{i=0}^{\infty} \hat{g}_i(x, m) \tag{III.23}$$

where $m = kN + s$,

$$\hat{g}_0(x, m) = w^{(k-1)N} q_+(x, m)$$

and

$$\hat{g}_i(x, m) = \sum_{n=m+1}^{\infty} w^{m-n} k_2(x, n, m)\, \hat{g}_{i-1}(x, n).$$

Since $|w| \geqslant 1$ on H, it follows from Lemma 4 that,

$$|\hat{g}_0(x, m)| < |w^{s+2}\hat{g}_0(x, m)| < D, \qquad m = kN + s.$$

Now using (III.16), (III.7), Lemma 3, the fact that the $a(i)$'s are strictly bounded away from zero, and the above inequality yields

$$|\hat{g}_1| < D \sum_{n=m+1}^{\infty} \hat{A} k_2(n) \frac{(n+N)}{N + |1 - w^{-2N}|\,(n+N)}$$

where again the fact that $|w^{-1}| \leqslant 1$ on H has been used. By induction one finds,

$$|\hat{g}_i| < D \frac{1}{i!} \left\{ \sum_{n=m+1}^{\infty} \hat{A} k_2(n) \frac{(n+N)}{N + |1 - w^{-2N}|\,(n+N)} \right\}^i$$

which upon substitution into (III.15) (with $n_0 = \infty$) gives

$$\begin{aligned} |w^{(k-1)N} p_+(x, m)| &= \left| \sum_{i=0}^{\infty} \hat{g}_i \right| \qquad\qquad \text{(III.24)} \\ &< D \exp\left\{ \hat{A} \sum_{n=m+1}^{\infty} k_2(n) \frac{(n+N)}{N + |1 - w^{-2N}|\,(n+N)} \right\}. \end{aligned}$$

Since each of the $\hat{g}_i$ are analytic and single valued on G and continuous on H, (III.20) and (III.24) imply that (III.23) converges uniformly on closed subsets of $H \backslash (w^{2N} = 1)$. Consequently, $w^{(k-1)N} p_+(x, m)$, $m = kN + s$ is analytic and single valued on G and continuous on $H \backslash (w^{2N} = 1)$. If (III.22) holds then (III.23) converges uniformly on H giving the continuity of $w^{(k-1)N} p_+(x, m)$ on H. Subtracting $q_+(x, m)$ from both sides of (III.15) (after setting $n_0 = \infty$) and then taking magnitudes and using (III.24) shows that (III.21) converges uniformly on closed subsets of $H\setminus(w^{2N}=1)$ if one has (III.20), while (III.21) converges uniformly on H if one has (III.22).

COROLLARY 3. *If* (III.20) *holds then*

$$\lim_{n_0 \to \infty} |w^{(k-1)N}(\hat{p}_+(x, m) - \hat{p}_+(x, m, n_0))| = 0, \qquad m = kN + s \tag{III.25}$$

uniformly on closed subsets of $H\setminus(w^{2N}=1)$. *If* (III.22) *holds then* (III.25) *converges uniformly on* H.

Proof. Subtracting the integral equation for $p_+(x, m)$ from (III.15) yields

$$\begin{aligned} w^{(k-1)N}&(\hat{p}_+(x, m) - \hat{p}_+(x, m, n_0)) \\ &= \sum_{n=n_0+1}^{\infty} k_2(x, n, m)\, w^{(k-1)N}\hat{p}_+(x, n) \\ &\quad + \sum_{n=m+1}^{n_0} k_2(x, n, m)\, w^{(k-1)N}(\hat{p}_+(x, n) - \hat{p}_+(x, n, n_0)). \end{aligned}$$

The method of successive approximations now gives

$$\begin{aligned} |w^{(k-1)N}&(\hat{p}_+(x, m) - \hat{p}_+(x, m, n_0))| \\ &\leqslant \sum_{n=n_0+1}^{\infty} \frac{k_2(n)(n+N)}{N + |1 - w^{-2N}|\,(n+N)} |w^{(k-1)N}\hat{p}_+(x, n)| \\ &\quad \times \exp \hat{A} \left\{ \sum_{l=m+1}^{\infty} k_2(l) \frac{(l+N)}{N + |1 - w^{-2N}|\,(l+N)} \right\} \end{aligned}$$

from which the conclusions of the corollary follow.

COROLLARY 4. *Let* $H' = G' \cup \partial G'$, *if* (III.20) *holds then there exists a solution of* (III.1) *such that*

$$\lim_{n \to \infty} |w^{(1-k)N}(\hat{p}_-(x, n) - q_-(x, n))| = 0$$

uniformly on closed sets of $H'\setminus(w^{2N}=1)$. *If* (III.22) *holds the convergence is uniform on* H'. *On* E, $p_-(x, n) = \overline{p_+(x, n)}$.

Proof. Letting $w^N \to w^{-N}$, $q_+(x, m) \to q_-(x, m)$, and $p(x, m) \to p_-(x, m)$ in the above discussion gives the first two assertions of the corollary. The third follows from integral equations satisfied by $p_+(x, n)$ and the facts that on E, $w^N = \bar{w}^{-N}$ and $q_-(x, n) = \overline{q_+(x, n)}$.

LEMMA 7. *Given* (III.20) *then for* $x \notin E$,

$$a(n+1)[\overline{p_+(x, n+1)}\, p_+(x, n) - p_+(x, n+1)\, \overline{p_+(x, n)}]$$

$$= (x - \bar{x}) \sum_{i=n+1}^{\infty} |p_+(x, i)|^2, \tag{III.26}$$

$$a(n+1)[p_+(x, n+1)\, p_+(x, n)' - p_+(x, n+1)'\, p_+(x, n)]$$

$$= \sum_{i=n+1}^{\infty} p_+(x, i)^2. \tag{III.27}$$

For $x \in E \backslash (w^{2N} = 1)$, *one finds*

$$p(x, n) = \frac{f_-(x)\, p_+(x, n) - f_+(x)\, p_-(x, n)}{a_0(0)\, q(x, N-1)[w^{-N} - w^N]} \tag{III.28}$$

where

$$f_\pm(x) = W[p, p_\pm] = a(0)\, p_\pm(x, -1). \tag{III.29}$$

Proof. Since $p_+(x, n)$ satisfies (III.1), (III.26) and (III.27) follow from routine manipulations and the facts that $\lim_{n \to \infty} p_+(x, n) = 0$ and $\lim_{n \to \infty} p_+(x, n)' = 0$ for $x \notin E$. To show (III.28) we note that $p_+(x, n)$ and $p_-(x, n)$ are solutions of (III.1) that are continuous for $x \in E$. Since $W[p_-, p_+] = a_0(0)\, q(x, N-1)[w^{-N} - w^N]$, we see that p_- and p_+ are linearly independent for $x \in E \backslash (w^{2N} = 1)$. Writing $p(x, n) = Ap_+(x, n) + Bp_-(x, n)$ one finds

$$A = \frac{W[p, p_+]}{W[p_-, p_+]} \quad \text{and} \quad B = -\frac{W[p, p_-]}{W[p_-, p_+]}$$

which yields the results.

We now divide the zeros of $p_+(z, n)$ into two categories: category $R_1(n)$ contains all the roots of $p_+(z, n)$ that are also zeros of $p_+(z, n-1)$, while $R_2(n)$ contains all the other zeros of $p_+(z, n)$.

LEMMA 8. *Suppose* (III.20) *holds, then all the zeros of* $p_+(x, n)$ *in* $G - \{\infty\}$ *are real, and* $p_+(x, n) \neq 0$ *for* $x \in E \backslash (w^{2N} = 1)$. *If* $x_1 \in R_2(n)$ *and* $x_1 \in G$, *then* x_1 *is a simple zero of* $p_+(x, n)$. *Between two consecutive zeros of* $p_+(x, n)$ *belonging to* $R_2(n)$ *that are not separated by an interval of* E *there is a zero of* $p_+(z, n-1)$ *and a zero of* $p_+(z, n+1)$.

Proof. From (III.26) one finds that all the zeros in $R_2(n)$ are real and from (III.27) simple. Let x_1 and $x_2 \in R_2(n)$ be two consecutive zeros of $p_+(z, n)$ such that an interval of E does not lie between them. Then the sum in (III.27) is positive at x_1 and x_2. Since $p_+(x, n)$ is real for all $x_1 \leqslant x \leqslant x_2$, $p_+(x, n)'$ must change signs between x_1 and x_2 implying through (III.27) that $p_+(x, n-1)$ and $p_+(x, n+1)$ change signs. Since $p_+(x, n-1)$ and $p_+(x, n+1)$ are real for $x_1 \leqslant x \leqslant x_2$ they each must have a zero inside that interval. To show that the zeros in $R_1(n)$ are real, we begin by noting that if $x_1 \in R_1(n)$, $x_1 \in R_1(i)$ $\forall i$. Now consider the system of polynomials satisfying (III.1) and (III.14). Since $p_+(z, m, n_0) = q_+(z, m)$ for $m > n_0$, $x \in R_1^{n_0}(n)$ only if it is a common zero of $q_+(z, i)$, $i > n_0$. But from (II.24) any common zero of $q_+(z, i)$ must be a zero of $q(x, N-1)$ which is real, implying all the zeros in $R_1^{n_0}(n)$ are real. This coupled with the arguments above show that all the zeros of $p_+(z, m, n_0)$ are real. The reality of the zeros of $p_+(z, n)$ now follows from Corollary 3 and Hurwitz's theorem. That $p_+(x, n) \neq 0$ for $x \in E \backslash (w^{2N} = 1)$ is a consequence of (III.28). For if we replace $p(x, n)$ by $p^{(k)}(x, n)$ we must replace $p_\pm(x, -1)$ by $p_\pm(x, k-1)$. Therefore if $p_+(x, k-1)$ has a zero for $x \in E \backslash (w^{2N} = 1)$ then so would $p_-(x, k-1)$ by Corollary 4 implying that all the $p^{(k)}(x, m)$ would have a zero at that point contradicting the well-known interlacing property of their zeros.

Let $J_{l_2 \to l_2}$ be the infinite dimensional matrix representation of (III.1). Then x_i is an eigenvalue of J iff there exists a nonzero $\psi \in l_2$ such that $J\psi = x_i \psi$.

LEMMA 9. *Suppose* (III.20) *holds. If* (a) $f_+(x_1) = 0$, $x_1 \in R_2(-1)$, *and* $x_1 \notin E$, *or* (b) $f_+(x_1) = 0$, $f'_+(x_1) = 0$, $x_1 \in R_1(-1)$ *and* $x_1 \notin E$, *then* x_1 *is an eigenvalue of* J *and*

$$p(x_1, n) = \frac{\tilde{p}_+(x_1, n)}{\tilde{p}_+(x_1, 0)} \tag{III.30}$$

where

$$\begin{aligned} \tilde{p}_+(x, n) &= p_+(x, n) && \text{if } a \text{ holds,} \\ &= p_+(x, n)/(x - x_1) && \text{if } b \text{ holds.} \end{aligned} \tag{III.31}$$

Proof. We begin with case (a) and consider the vector $\psi = \{p_+(x_1, n)\}_{n=0}^{\infty}$. Then $\psi \in l_2$, $\psi \not\equiv 0$ and $J\psi = x_1 \psi$ showing that x_1 is an eigenvalue of J. In the case of (b) we note that $p_+(x_1, n) = 0$ and $q_+(x_1, n) = 0$ for all n. Consequently we can divide $p_+(x, n)$ by $x - x_1$ and not change equations (III.1) or (III.15). Since

$$\left. \frac{q_+(x, nN-1)}{x - x_1} \right|_{x = x_1} \neq 0,$$

it follows that for large enough n,

$$\left.\frac{p_+(x, nN-1)}{x-x_1}\right|_{x=x_1} \neq 0.$$

Consequently, the vector $\psi = \{\tilde{p}_+(x_1, n)\}_{n=0}^{\infty}$ is an eigenvector of J with eigenvalue x_1.

THEOREM 4. *If* (III.22) *holds then $p_+(x, n)$ has a finite number of zeros on H.*

Proof. We need only consider the number of zeros of $p_+(x, n)$ on $E^* \cup (w^{2N}=1)$, since on $E \backslash (w^{2N}=1)$ $p_+(x, n)$ has no zeros, while for $x \in \mathbb{R}$, $x \notin E^* \cup E$, $p_+(x, n)$ $n=-1, 0, 1,...$ has only a finite number of zeros (Geronimo [9, Theorem III.1]). Consider now an interval $E^*_{i_0}$ of E^* and let x_1 and x_2, $x_1 < x_2$ be the end points. We suppose there exists a $q_+(x, m)$ such that $q_+(x_1, m) \neq 0$. For if this is not the case then $x_1 \in R_1^0(m)$, where 0 denotes the periodic system, and $q_+(x_1, j) = 0$ $\forall j$. Consequently we may divide the integral equation for $p_+(x, n)$ ((III.15) after setting $n_0 = \infty$) by $(x-x_1)$ and use the same manipulations that led to (III.21) to show that $\lim_{m\to\infty} |w^{(k-1)N}(\hat{p}^*_+(x, m) - q^*_+(x, m)| = 0$, $m = kN+s$ uniformly on H. Here $\hat{p}^*_+(x, m) = \hat{p}_+(x, m)/(x-x_1)$, $q^*_+(x, m) = q_+(x, m)/(x-x_1)$. If there does not exist a $q^*_+(x, m)$ such that $q^*_+(x_1, m) \neq 0$ one repeats the above procedure once again. (III.31) shows that this procedure will be necessary at most two times. Suppose that $q_+(x_1, m_0) \neq 0$, then (II.27) shows that $q_+(x_1, m_0+jN) \neq 0$. Let $x_0 \in (\frac{3}{4}(x_1+x_2), x_2)$ such that $q_+(x_0, m) \neq 0$ $\forall m$, and let D_{x_1,x_0} be the open disk centered on the real axis with x_1 and x_0 on its boundary. Writing $m_0 = k_0 N + s_0$ we see from (III.21) and (II.27) that there exists a j_0 such that for all $j \geqslant j_0$,

$$\begin{aligned} &|w^{(k_0+j-1)N}(\hat{p}_+(x, m_0+jN) - q_+(x, m_0+jN))| \\ &\quad \leqslant |w^{(k_0+j-1)N} q_+(x, m_0+jN)|, \end{aligned} \tag{III.32}$$

$x \in \bar{D}_{x_1,x_0}$. Consequently, by Rouché's theorem $p_+(x, m_0+jN)$ for $j \geqslant j_0$ has the same number of zeros inside D_{x_1,x_0} as $q_+(x, m_0+jN)$, i.e., a finite number N_0. Lemma 8 now tells us that $p_+(x, n)$ $\forall n$ has a finite number of zeros in D_{x_1,x_0} that does not exceed some number N_1. One now repeats the above argument for the open disk $D_{\hat{x}_0,x_2}$, where $\hat{x}_0 \in (x_1, (x_1+x_2)/4)$ and $q_+(\hat{x}_0, m) \neq 0$ $\forall m$. This shows that $p_+(x, n)$ has a finite number of zeros in the interval $[x_1, x_2]$. Repeating the above argument for the other at most $N-2$ intervals of E^* gives the result.

We now return to the system of polynomials satisfying (III.1) and (III.14).

LEMMA 10. *If* $q(x_j, N-1)=0$ *and* $\rho(x_j)=0$ *then* $p_\pm(x_j, m, n_0)=0$ *for all m. Consequently,* $(w^{-N}-w^N)/f_+(x, n_0)$ *is continuous on* ∂G*. Furthermore,*

$$\begin{aligned} f_+(x, n_0) f_-(x, n_0) = a_0(0)\, q(x, N-1) \\ \times\ [a_0(n_0)\, p(x, n_0+N-1, n_0)\, p(x, n_0, n_0) \\ - a(n_0)\, p(x, n_0-1, n_0)\, p(x, n_0+N, n_0)]. \end{aligned} \tag{III.33}$$

Proof. If $q(x_j, N-1)=0$ and $\rho(x_j)=0$, then by Lemma 5 $q_+(x_j, i)=0$ for all i. But then $x_j \in R_1^{n_0}(n)$ which implies that $p_+(x_j, m, n_0)=0$ for all m. Since $q_-(x, m)=q_+(x, m)$ on E the result follows for $p_-(x, m, n_0)$. To show the second part of the lemma we note that from (III.28) one has for $x \in \partial G \backslash (w^{2N}=1)$

$$\frac{w^{-N}-w^N}{f_+(x, n_0)}\, p(x, m, n_0) = \frac{S(x)\, p_+(x, m, n_0) - p_-(x, m, n_0)}{a_0(N)\, q(x, N-1)} \tag{III.34}$$

where

$$S(x) = \frac{f_-(x, n_0)}{f_+(x, n_0)}.$$

From the definition of p_- and p_+ one finds that $|S(x)|=1$, $x \in \partial G \backslash (w^{2N}=1)$. Since

$$\frac{p_+(x, m, n_0)}{q(x, N-1)},$$

$$\frac{p_-(x, m, n_0)}{q(x, N-1)}, \qquad f_+(x, n_0) \qquad \text{and} \qquad f_-(x, n_0)$$

are continuous on ∂G, $(w^{-N}-w^N)/f_+(x, n_0)$ is continuous there also. To find (III.33) we note that $p_-(x, m, n_0)$ can be analytically continued onto G. Therefore we can use (III.28) on G. Setting $n=n_0$ and $n=n_0+N$ in (III.28) then multiplying by $p_+(x, n_0+N, n_0)$ and $p_+(x, n_0, n_0)$ respectively and subtracting yields

$$\begin{aligned} q_+(x, n_0)[p(x, n_0+N, n_0) - w^{-N} p(x, n_0, n_0)] \\ = \frac{f_+(x, n_0)\, q_+(x, n_0)\, q_-(x, n_0)}{a_0(N)\, q(x, N-1)}. \end{aligned}$$

Here we have used the facts that $p_{\pm}(x, n_0+n, n_0) = q_{\pm}(x, n_0+n)$ and $q_{\pm}(x, n_0+N) = w^{\mp N} q_{\pm}(x, n_0)$. Now using (II.31) yields

$$f_+(x, n_0) = \frac{a_0(n_0+1) q_+(x, n_0)}{q^{(n_0+1)}(x, N-1)} [p(x, n_0+N, n_0) - w^{-N} p(x, n_0, n_0)]. \quad \text{(III.35)}$$

Consequently,

$$\begin{aligned} f_+(x, n_0) f_-(x, n_0) = {} & \frac{a_0(n_0+1)^2}{q^{(n_0+1)}(x, N-1)} q_+(x, n_0)\, q_-(x, n_0) \\ & \times [p(x, n_0+N, n_0)^2 + p(x, n_0, n_0)^2 - (w^N + w^{-N}) \\ & \times p(x, n_0+N, n_0)\, p(x, n_0, n_0)]. \end{aligned} \quad \text{(III.36)}$$

Since $p(x, n, n_0)$, $p(x, n+N, n_0)$, and $q^{(n_0+1)}(x, n-n_0-1)$ satisfy the same recurrence formula for $n \geqslant n_0$, one can write $p(x, n+N, n_0) = Ap(x, n, n_0) + Bq^{(n_0+1)}(x, n-n_0-1)$. Setting $n = n_0$ gives A, and setting $n = n_0+1$ gives B. Now setting $n = n_0+N$ yields

$$q^{(n_0+1)}(x, N-1) = \frac{p(x, n_0+N, n_0)^2 - p(x, n_0, n_0)\, p(x, n_0+2N, n_0)}{p(x, n_0+N, n_0)\, p(x, n_0+1, n_0) - p(x, n_0, n_0)\, p(x, n_0+N+1, n_0)}.$$

Substituting (II.31) into (III.36) and then substituting the above equation into (III.36) yields

$$\begin{aligned} f_+(x, n_0) f_-(x, n_0) = {} & a_0(0)\, a_0(n_0+1)\, q(x, N-1) \\ & \times [p(x, n_0+N, n_0)\, p(x, n_0+1, n_0) \\ & - p(x, n_0, n_0)\, p(x, n_0+N+1, n_0)] \end{aligned}$$

where the fact that $p(x, n_0+2N, n_0)$, $p(x, n_0+N, n_0)$, and $p(x, n_0, n_0)$ satisfy (II.8) has been used. Now incrementing $p(x, n_0+1, n_0)$ and $p(x, n_0+N+1, n_0)$ down by one using (III.1) gives (III.33).

IV. Construction of the Measure

We now proceed to construct the measure associated with the three term recurrence formula (III.1) whose coefficients satisfy (III.3). We will begin by considering the systems satisfying (III.1) and (III.14).

THEOREM 5. *Suppose the coefficients in* (III.1) *satisfy* (III.14), *then the* *measure μ with respect to which the polynomials $p(x, m, n_0)$ are orthonormal can be written as*

$$d\mu = \sigma\, dx + \sum_{i=1}^{M} \rho_i \delta(x - x_i) \tag{IV.1}$$

where

$$\sigma(x) = \frac{1}{2\pi |f_+(x, n_0)|^2} \left[a_0(N)\, q(x, N-1) \times \left(4 - \left\{ q(x, N) - \frac{a_0(N)}{a_0(N+1)} q^{(1)}(x, N-2) \right\}^2 \right)^{1/2} \right]$$

$$x \in E \backslash (w^{2N} = 1), \tag{IV.2}$$

and

$$\rho_k = \frac{\tilde{p}_j(x_k, 0, n_0)}{\tilde{f}(x_k, n_0)'}. \tag{IV.3}$$

Here E is given by (II.20) *and $\tilde{p}_+(x, n)$, $n = -1, 0, \ldots$, by* (III.31). *x_k is such that $x_k \in E^*$ and $\tilde{f}(x_k, n_0) = a(0)\, \tilde{p}_+(x_k, -1, n_0) = 0$.*

Proof. Consider the contour $\Gamma = \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_1 = \{z; |z| = 1, z \neq e^{\pm(i\pi/N)t}, t = 1, 2, \ldots, N-1\}$$

and Γ_2 is the union of $2N$-2 contours that encircles the $2N - 2$ images of E^* under w^{-1} (see Fig. 3).

Now consider the integral ($m \leqslant n$),

$$I = -\int_{\Gamma_1} - \int_{\Gamma_2} \frac{p(x, m, n_0)\, p_+(x, n, n_0)}{2\pi i f_+(x, n_0)} h'(z)\, dz = I_1 + I_2, \tag{IV.4}$$

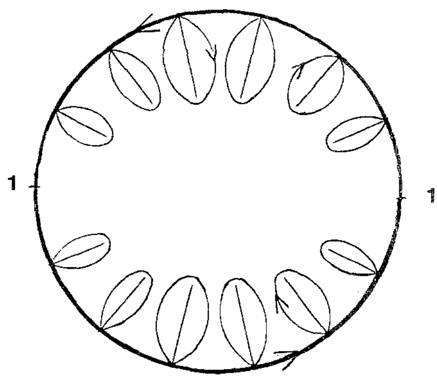

FIG. 3. The contour Γ. The set Γ_1 is indicated in thick lines, the contours Γ_2 are in thin lines. The arrows indicate the direction ($N = 7$).

where $h(z)$ is the inverse mapping of w^{-1}, i.e., $z=w^{-1}(x)$ and $x=h(z)$. Since from (II.15),

$$h'(z)=1\Big/\frac{dw^{-1}}{dx}\Big|_{x=h(z)}$$
$$=-N(w^{-N}-w^{N})\Big/\left(w^{-1}\left\{q'(x,N)-\frac{a_0(N)}{a_0(N+1)}q^{(1)}(x,N-2)'\right\}\Big|_{x=h(z)}\right), \tag{IV.5}$$

it follows from Lemma 10 that I is well defined. We first consider I_1. Using (III.28) to eliminate $p_+(x,n,n_0)$ and using the fact that $f_+(x,n_0)=\overline{f_-(x,n_0)}$ on Γ_1 yields

$$I_1=\frac{1}{2\pi i}\int_{\Gamma_1}\frac{p(x,m,n_0)\,p_-(x,n,n_0)\,h'(z)\,dz}{f_-(x,n_0)}$$
$$+\frac{-1}{2\pi i}\int_{\Gamma_1}\frac{W[p_+,p_-]\,p(x,m,n_0)\,p(x,n,n_0)\,h'(z)\,dz}{|f_+(x,n_0)|^2}. \tag{IV.6}$$

Writing $z=e^{i\theta}$ in the first integral in the above equation then letting $\theta\to-\theta$ and using the fact that under this change of variables, $p_-(x,n,n_0)\to p_+(x,n,n_0)$ for all n while by (IV.5) $e^{i\theta}h'(e^{i\theta})\to -e^{-i\theta}h'(e^{-i\theta})$ gives

$$I_1=\frac{-1}{4\pi i}\int_{\Gamma_1}\frac{W[p_+,p_-]\,p(x,m,n_0)\,p(x,n,n_0)\,h'(z)\,dz}{|f_+(x,n_0)|^2}.$$

Writing $\Gamma_1=\Gamma_+\cup\Gamma_-$, where $\Gamma_+=\Gamma_1\cap\{z\colon \operatorname{Im} z\geqslant 0\}$ and $\Gamma_-=\Gamma_1\cap\{z\colon \operatorname{Im} z\leqslant 0\}$ then performing the same operation on the integral of Γ_- that one used on the first integral in (IV.6) yields

$$I_1=\frac{-1}{2\pi i}\int_{\Gamma_+}\frac{W[p_+,p_-]\,p(x,m,n_0)\,p(x,n,n_0)\,h'(z)\,dz}{|f_+(x,n_0)|^2}.$$

Now mapping back to the set E and using the fact that on Γ_+

$$W[p_+,p_-]=-ia_0(N)\,q(x,N-1)$$
$$\times\sqrt{4-\{q(x,N)-(a_0(N)/a_0(N+1))\,q^{(1)}(x,N-2)\}^2}$$

gives

$$I_1=\int_E p(x,m,n_0)\,p(x,n,n_0)\,\sigma(x)\,dx \tag{IV.7}$$

where $\sigma(x)$ is given by (IV.2). Now Γ is a closed contour and one can evaluate the integral using the residue theorem. The possible singularities will be at $z=0$ and the zeros of $f_+(x, n_0)$ that are inside Γ. For large x one has that

$$p(x, m, n_0) \approx \left(\prod_{j=1}^{m} \frac{1}{a(j)} \right) x^m,$$

while from (II.24), (II.32), and (II.15) one finds that

$$\frac{p_+(x, n, n_0)}{f_+(x, n_0)} \approx \prod_{j=1}^{n} a(j)\, x^{-n-1}.$$

Now $z = w(x)^{-1} \approx Cx^{-1}$, where $C = (\prod_{j=1}^{N} a_0(j))^{1/N}$ is the capacity of E. Consequently,

$$\frac{p(x, m, n_0)\, p_+(x, n, n_0)\, h'(z)}{f_+(x, n_0)} \approx -\prod_{j=1}^{m} \frac{1}{a(j)} \prod_{i=1}^{n} a(i)\, C^{m-n} z^{n-m-1} \qquad \text{(IV.8)}$$

where the fact that $h'(z) \sim -C/z^2$ has been used. Therefore one sees that the residue at $z=0$ is $-\delta_{n,m}$, $m \leqslant n$. To evaluate the other residues we note that all the zeros of $p_+(x, -1, n_0)$ inside Γ are in $R_2^{n_0}(-1)$. Hence using (III.30), one finds

$$I = \delta_{n,m} - \sum_i \frac{p(h(z_i), m, n_0)\, p(h(z_i), n, n_0)\, p_+(h(z_i), 0, n_0)}{(d/dz) f_+(h(z), n_0)\,|_{z=z_i}}\, h'(z_i) \qquad \text{(IV.9)}$$

where $f_+(h(z_1), n_0) = 0$ and the sum is a finite sum. In order to evaluate the contribution due to Γ_2 we make the change of variable $z \to x = h(z)$. The contour Γ_1 is mapped to the intervals E (circumscribed twice), while the contour Γ_2 is mapped to $N-1$ contours, each contour containing one component of E^* (see Fig. 4). Therefore

$$\frac{1}{2\pi i} \int_{\Gamma_2} \frac{p(x, m, n_0)\, p_+(x, n, m_0)}{f_+(x, n_0)}\, h'(z)\, dz$$
$$= \frac{1}{2\pi i} \sum_{j=1}^{N-1} \int_{D_j} \frac{p(x, m, n_0)\, p_+(x, n, n_0)}{f_+(x, n_0)}\, dx$$

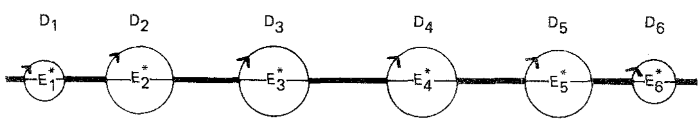

FIG. 4. The images of the contours Γ and Γ_2 through the mapping $h(z)$. The thick line is the set E and the small contours are the images of Γ_2 ($N=7$).

where D_j is the contour around E_j^*. The only zeros of $f_+(x, n_0)$ that will contribute are those in $R_2^{n_0}(-1)$ and those in $R_1^{n_0}(-1)$ such that $f'_+(x_1, n_0)=0$, $x_1 \in R_1^{n_0}(-1)$. Consequently,

$$\frac{1}{2\pi i}\int_{\Gamma_2} \frac{p(x, m, n_0)\, p_+(x, n, m_0)}{f_+(x, n_0)}\, h'(z)\, dz$$

$$= -\sum_k p(x_k, m, n_0)\, p(x_k, n, n_0) \frac{\tilde{p}_+(x_k, 0, n_0)}{\tilde{f}'(x_k, n_0)} \tag{IV.10}$$

where the minus sign comes from the fact that one is going around the contours D_j in the clockwise direction. Changing variables in (IV.9) then substituting the result along with (IV.7) and (IV.10) into (IV.4) gives the theorem.

One may eliminate the $a_0(N)\, q(x, N-1)$ in (IV.2) using (III.33) to obtain the result found by Geronimus [16]. Furthermore, in this case we may analytically continue $p_-(x, n, n_0)$ on to G and evaluate $p_+(x, n, n_0)$ at a zero of $f_+(x, n_0)$ using (III.28). Supposing $f_+(x_k, n_0)=0$ and $q(x_k, N-1)\neq 0$ one finds that

$$\rho_k = \frac{1}{f'_+(x_k, n_0) f_-(x_k, n_0)} \left\{ a_0(0)\, q(x_k, N-1) \right.$$

$$\left. \times \sqrt{\left[q(x_k, N) - \frac{a_0(N)}{a_0(N+1)}\, q^{(1)}(x_k, N-2) \right]^2 - 4} \right\}$$

Eliminating $f_+(x, n_0) f_-(x, n_0)$ using (III.33) gives the formula for the mass obtained by Geronimus [16]. Geronimus [17] has also obtained

COROLLARY 11. *Let μ be the measure associated with* (III.3) *then* $\mu(x) = \lim_{n_0\to\infty} \mu_{n_0}(x)$ *and* $\mu(x) = \mu_c(x) + \mu_d(x)$, *where* $\mu_c(x)$ *is continuous nondecreasing function whose points of increase are dense in* E, *and* μ_d *is a jump function. Let* E_1 *contain all the points of discontinuity of* μ_d *and let* E'_1 *be the derived set of* E_1, *then* $E'_1 \subset E$.

Proof. That $\mu_{n_0}(x) \to \mu(x)$ follows from Helly's theorem and the uniqueness of the moment problem. To show the second part let J and J_0 be the infinite Jacobi matrices given by (III.1) and (II.2) respectively and set $J = J_0 + J_p$, where $J_p = J - J_0$. By (III.3) J_p is a compact operator and it is a consequence of a theorem of Weyl [26] that the essential spectrum of J is the same as the essential spectrum of J_0 thus giving the result.

THEOREM 6. *If* (III.20) *holds then*

$$d\mu = \sigma(x)\, dx + \sum_i \rho_i \delta(x - x_1) \tag{IV.11}$$

where

$$\sigma(x)=a_0(N)\,q(x,N-1)\times\frac{(4-\{q(x,N)-(a_0(N)/a_0(N+1))\,q^{(1)}(x,N-2)\}^2)^{1/2}}{2\pi\,|f_+(x)|^2} \tag{IV.12}$$

with $x\in E\backslash(w^{2N}=1)$ *and*

$$\rho_i=\frac{\tilde{p}_+(x_i,0)}{\tilde{f}_+(x_i)'} \tag{IV.13}$$

for $x_i\in E^c$. *If* (III.22) *holds then the sum over i is finite.*

Proof. The theorem follows from Theorems 4 and 5, Corollary 3, and Lemma 8.

THEOREM 7. *Suppose* $\sum_{n=0}^{\infty}\ln(n+N+1)\,k_2(n)<\infty$ *then* $\ln\sigma(x)\in L^1(\mu_e)$, *where* μ_e *is the equilibrium measure on E, i.e.,*

$$\begin{aligned}\mu_e(B)&=\frac{1}{N\pi}\int_B\left|\frac{q(x,N)'-(a_0(N)/a_0(N+1))\,q'^{(1)}(x,N-2)}{\sqrt{4-\left\{q(x,N)-\dfrac{a_0(N)}{a_0(N+1)}\,q^{(1)}(x,N-2)\right\}^2}}\right|dx\\&=\int_B\Delta(x)\,dx;\end{aligned}$$

B a Borel subset of E.

Proof. From Jensens' theorem one finds

$$\int_E\ln^+\frac{\sigma(x)}{\Delta(x)}\,d\mu_e\leqslant\int_E\sigma(x)\,dx<\infty$$

and

$$\int_E\ln^+\frac{1}{\Delta(x)}\,d\mu_e\leqslant\int_E dx<\infty$$

which implies $\int_E\ln^+\sigma(x)\,d\mu_e(x)<\infty$.

We must now show that $\int_E\ln^+(1/\sigma(x))\,d\mu_e<\infty$. Consider the integral

$$I=\int_E\ln^+|f_+(x,n_0)|\,d\mu_e$$

where $f_+(x, n_0)$ is given by (III.35). Substituting (III.24) into the above equation and observing that $\ln^+ \exp g(x) = g(x)$ if $g(x) \geqslant 0$ yields

$$I \leqslant C + A \int_E \left(\sum_{n=0}^{n_0} k_2(n) \frac{(n+N)}{N + |1 - w^{-2N}|\,(n+N)} \right) d\mu_e.$$

Since $\int_E f(x)\, d\mu_e = \int_0^{2\pi} f(h(e^{i\theta}))(d\theta/2\pi)$ (Sario and Nakai [22]) we find

$$I \leqslant C + A \sum_{n=0}^{n_0} k_2(n) \int_0^{2\pi} \frac{(n+N)}{N + |1 - e^{i2N\theta}|\,(n+N)} \frac{d\theta}{2\pi}$$

which gives

$$I \leqslant C + AC_1 \sum_{n=0}^{n_0} \ln(n+N+1)\, k_2(n).$$

Now letting $n_0 \to \infty$ and then using Corollary 3 and (IV.12) gives the result.

V. Asymptotic Behavior

We begin this section by deriving a formula first obtained by Geronimus [16].

Lemma 11. *Let $p_1(x, n)$ be any solution of* (III.1) *then*

$$\begin{aligned} p_1(x, n+2N)\, p^{(n+1)}(x, N-1) &= p_1(x, n+N)\, p^{(n+1)}(x, 2N-1) \\ &\quad - \frac{a(n+1)}{a(n+N+1)} p_1(x, n)\, p^{(n+N+1)}(x, N-1). \end{aligned} \tag{V.1}$$

Proof. We will begin by showing that the above formula is true for $p(x, n)$. Since $p(x, n)$, $p^{(m)}(x, n-m)$, and $p^{(m+1)}(x, n-m-1)$ are solutions of (III.1) one finds

$$\begin{aligned} p(x, n) &= p(x, m)\, p^{(m)}(x, n-m) \\ &\quad - \frac{a(m)}{a(m+1)} p(x, m-1)\, p^{(m+1)}(x, n-m-1). \end{aligned} \tag{V.2}$$

Multiplying the above equation with $n = n+N$ and $m = n+1$, and with

$n=n+2N$ and $m=n+1$ by $p^{(n+1)}(x, 2N-1)$ and $p^{(n+1)}(x, N-1)$ respectively and then subtracting the resulting equations yields

$$\begin{aligned} p(x, n+N)\, p^{(n+1)}(x, 2N-1) - p(x, n+2N)\, p^{(n+1)}(x, N-1) \\ = \frac{a(n+1)}{a(n+2)} p(x, n)\{p^{(n+2)}(x, 2N-2)\, p^{(n+1)}(x, N-1) \\ - p^{(n+2)}(x, N-2)\, p^{(n+1)}(x, 2N-1)\}. \end{aligned} \tag{V.3}$$

By analogy with (V.2) one has

$$\begin{aligned} p^{(n+m)}(x, 2N-m) = p^{(n+m)}(x, N-m)\, p^{(n+N)}(x, N) \\ - \frac{a(N+n)}{a(N+n+1)} p^{(n+m)}(x, N-m-1) \\ \times p^{(n+N+1)}(x, N-1). \end{aligned} \tag{V.4}$$

Multiplying the above equation with $m=1$, and with $m=2$ by $p^{(n+2)}(x, N-2)$ and $p^{(n+1)}(x, N-1)$ respectively and then subtracting the resulting equations yields

$$\begin{aligned} p^{(n+2)}(x, 2N-2)\, p^{(n+1)}(x, N-1) - p^{(n+1)}(x, 2N-1)\, p^{(n+2)}(x, N-2) \\ = \frac{a(n+2)}{a(N+n+1)} p^{(n+N+1)}(x, N-1). \end{aligned} \tag{V.5}$$

In the above equation the fact that $W[p^{(n+1)}, p^{(n+2)}] = -a(n+2)$ has been used. Inserting (V.5) into (V.3) gives (V.1) for $p_1(x,n)=p(x,n)$. Using similar arguments one finds that $p^{(1)}(x, n+2N-1)$, $p^{(1)}(x, n+N-1)$, and $p^{(1)}(x, n-1)$ satisfy (V.1) and, since all solutions of (III.1) can be written as a linear combination of $p(x, n)$ and $p^{(1)}(x, n)$, the lemma is proved.

It is possible to convert (V.1) into two recurrence relations whenever (III.3) holds. The following lemma is a generalization of a result given by Geronimo and Case [8].

LEMMA 12. *Equations* (III.1) *and* (III.2) *with recurrence coefficients that satisfy* (III.3) *are equivalent to the two following relations*

$$\begin{aligned} p(x, n+N) = \frac{a_0(n+1)}{a(n+1)} \frac{p^{(n+1)}(x, N-1)}{q^{(n+1)}(x, N-1)} \\ \times \{(w^{\mp N} - B(n, x))\, p(x, n) + w^{\pm N} \psi^{\pm}(x, n)\} \end{aligned} \tag{V.6}$$

and

$$\psi^{\pm}(x, n+N) = \frac{a_0(n+1)}{a(n+1)} \frac{p^{(n+1)}(x, N-1)}{q^{(n+1)}(x, N-1)} \times \left\{ w^{\pm N}\psi^{\pm}(x, n) + \left[\left(1 - \left(\frac{a(n+1)}{a_0(n+1)} \times \frac{q^{(n+1)}(x, N-1)}{p^{(n+1)}(x, N-1)} \right)^2 \right) w^{\mp N} - B(n, x) \right] p(x, n) \right\}. \tag{V.7}$$

where

$$B(n, x) = w^N + w^{-N} - \frac{a(n+1)}{a_0(n+1)} q^{(n+1)}(x, N-1) \times \left\{ \frac{p^{(n)}(x, N)}{p^{(n+1)}(x, N-1)} - \frac{a(n)}{a(n+1)} \frac{p^{(n-N+1)}(x, N-2)}{p^{(n-N+1)}(x, N-1)} \right\}. \tag{V.8}$$

and $\psi^{\pm}(x, 0) = p(x, 0) = 1$.

Proof. Solve (V.6) for $\psi^{\pm}(x, n)$ and substitute this result into (V.7). The resulting equation is (V.1), where one replaces $p^{(n+1)}(x, 2N-1)$ by (V.4) with $m = 1$.

If one subtracts (V.6) from (V.7) one finds

$$\psi^{\pm}(x, n+N) = p(x, n+N) - \frac{a(n+1)}{a_0(n+1)} \frac{q^{(n+1)}(x, N-1)}{p^{(n+1)}(x, N-1)} w^{\mp N} p(x, n). \tag{V.9}$$

Consequently for the system satisfying (III.14) one finds, using (III.35), that

$$f_{\pm}(x, n_0) = a_0(n_0+1) \frac{q_+(x, n_0)}{q^{(n_0+1)}(x, N-1)} w^{\pm N} \psi^{\pm}(x, n_0). \tag{V.10}$$

This leads to

THEOREM 8. *Suppose that* (III.20) *is fulfilled, then for any integer j the following limit holds*:

$$\lim_{k \to \infty} w^{-kN} p(x, kN+j) = \frac{q^{(j+1)}(x, N-1) f_+(x)}{a_0(j+1)\, q_+(x, j)[w^N - w^{-N}]}$$

uniformly on closed subsets of G.

Proof. If (III.20) is fulfilled we can let $n_0 \to \infty$ in G to obtain

$$f_+(x) = \lim_{n_0 \to \infty} \frac{a_0(n_0+1)\, q_+(x, n_0)}{q^{(n_0+1)}(x, N-1)} w^N \psi^+(x, n_0). \tag{V.11}$$

Now setting $n_0 = kN + j$ in the above equation and using the properties of the periodic system one finds

$$f_+(x) = \frac{a_0(j+1)\, q_+(x, j)}{q^{(j+1)}(x, N-1)} \lim_{k \to \infty} w^{-(k-1)N} \psi^+(x, kN+j). \qquad \text{(V.12)}$$

If one subtracts the two equations given by (V.6) from each other and then multiplies the result by w^{-kN} one has

$$w^{-kN} p(x, kN+j) = \frac{w^{-(k-1)N} \psi^+(x, kN+j) - w^{-(k+1)N} \psi^-(x, kN+j)}{w^N - w^{-N}}. \qquad \text{(V.13)}$$

Consequently, the result will be proved if one can show that $\lim_{k \to \infty} w^{-(k+1)N} \psi^-(x, kN+j) = 0$ uniformly on closed subsets of G. Finding the analog of (III.8) for $p^{(n+1)}(x, N-1)$ shows that

$$\lim_{k \to \infty} \frac{\hat{p}^{(kN+j+1)}(x, N-1)}{q^{(j+1)}(x, N-1)} = 1$$

uniformly on closed subsets of $G \backslash \{\text{zeros of } q^{(j+1)}(x, N-1)\}$. Thus, from (V.9) and (III.10) we see that the result will be demonstrated if one can show that

$$\lim_{k \to \infty} w^{-(k+2)N} \hat{p}(x, (k+1)\, N+s) - w^{-(k+1)N} \hat{p}(x, kN+s) = 0$$

uniformly on closed subsets of G. Now from (III.8) we have

$$\begin{aligned} & w^{-(k+2)N} \hat{p}(x, (k+1)\, N+s) - w^{-(k+1)N} \hat{p}(x, kN+s) \\ &\quad = w^{-(k+2)N} q(x, (k+1)\, N+s) - w^{-(k+1)N} q(x, kN+s) \\ &\qquad + \sum_{n=0}^{kN+s-1} [w^{-(k+2)N} k_1(x, n, (k+1)\, N+s) \\ &\qquad - w^{-(k+1)N} k_1(x, n, kN+s)]\, \hat{p}(x, n) \\ &\qquad + \sum_{n=kN+s}^{(k+1)N+s-1} w^{-(k+2)N} k_1(x, n, (k+1)\, N+s)\, \hat{p}(x, n). \end{aligned} \qquad \text{(V.14)}$$

From (III.10) and (III.12) we see that the last term goes to zero as k tends to infinity uniformly on closed subsets of G. From (II.24) and (II.30) we find

$$\begin{aligned} |w^{-(k+2)N} q(x, (k+1)\, N+s) - w^{-(k+1)N} q(x, kN+s)| &= |w^{-(2k+2)N} q_+(x, s)| \\ &\leqslant D\, |w^{-2kN}|, \end{aligned} \qquad \text{(V.15)}$$

which leaves only the second term to be discussed. From (III.6) and (III.9) one finds

$$
\begin{aligned}
&w^{-(k+2)N}k_1(x, n, (k+1)N+s) - w^{-(k+1)N}k_1(x, n, kN+s) \\
&\quad = \frac{(b_0(n)-b(n))}{a_0(n+1)} w^{-(k+2)N}\{q^{(n+1)}(x, (k+1)N+s-n-1) \\
&\quad - w^N q^{(n+1)}(x, kN+s-n-1)\} \\
&\quad + \frac{a_0(n+1)}{a_0(n+2)}\left\{1-\frac{a^2(n+1)}{a_0^2(n+1)}\right\} w^{-(k+2)N}\{q^{(n+2)}(x, (k+1)N+s-n-2) \\
&\quad - w^N q^{(n+2)}(x, kN+s-n-2)\}. \qquad \text{(V.16)}
\end{aligned}
$$

Now setting $n = mM + p$ and using (II.22) and (II.23) yields

$$
\begin{aligned}
&|w^{-(k+2)N}k_1(x, n, (k+1)N+s) - w^{-(k+1)N}k_1(x, n, kN+s)| \\
&\quad \leqslant \left\{\left|\frac{b_0(n)-b(n)}{a_0(n-1)}\right| + \frac{a_0(n+1)}{a_0(n+2)}\left|1-\frac{a^2(n+1)}{a_0^2(n+1)}\right|\right\} Dw^{-kN}. \qquad \text{(V.17)}
\end{aligned}
$$

Substituting this result into the second term in (V.14) then using (III.10) gives the result.

On the spectrum we have, from (III.28), (III.21), Theorem 1 and the properties of $w(x)$ and $h(z)$.

THEOREM 8. *If* (III.20) *holds then for every* $x \in E \backslash (w^{2N} = 1)$

$$
\begin{aligned}
\lim_{k\to\infty} \Bigg[&\frac{p(x, kN+s)}{|q_+(x, s)|} \sqrt{8\pi\sigma(x)\, a_0(N)\, q(x, N-1)} \\
&\times \sqrt{\left[4 - \left\{q(x, N) - \frac{a_0(N)}{a_0(N+1)} q^{(1)}(x, N-2)\right\}^2\right]^{1/2}} \\
&- \cos(kN\theta + \Gamma(\theta, s)) \Bigg] = 0,
\end{aligned}
$$

where $\theta = \arg w(x)$ *and* $\Gamma(\theta, s) = -\arg f_+(h(e^{i\theta})) + \arg q_+(h(e^{i\theta}), s) + \pi/2$. *Furthermore the convergence is uniform on compact subsets of* $E \backslash (w^{2N} = 1)$.

ACKNOWLEDGMENTS

Both authors would like to thank the Division of Theoretical Physics at Saclay for the hospitality extended to them during their stay there.

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